1 Introduction

1.1 Control Charts

Manufactured products are often subject to variation from specifications. A process operating in a manner not subject to assignable causes is said to be in statistcal control. We shall be concerned with the study of control charts which provide a tool for the manufacturer to take action when products are no longer produced in accordance with the specification. The basic feature of the control chart consists of taking small samples from the production line and to calculate some statistic. The latter is then plotted on a graph which exhibits an upper and lower boundary for that such statistics when the process is in control. If the value of the statistic falls outside the control lines, there is a cause for concern. In what follows we shall construct control charts for the mean, the variance and the proportion.

2 Control Chart for the mean

Suppose that a random variable X has a normal distribution with mean $\mu$and variance σ^2. Then the probability that X will fall in the interval

$$[\mu - 3\sigma, \mu + 3\sigma]$$

is 0.9973. Put differently, on average only 27 observations out of $10,000$ will fall outside that interval. Of course, for most production processes, we do not know the underlying distribution of the variable being measured. However, the Central limit theorem asserts that for a large enough sample size, $\bar{X}_n$ will be approximately normally distributed with mean μ and variance $\frac{\sigma^2}{n}$ irrespective of the underlying distribution of X. The Central limit theorem forms the basis for the construction of control charts. Hence, if we take a subgroup of obervations from the production line and calculate the sample average $\bar{x}_n$ it should fall within the interval

$$\left[\mu - 3\frac{\sigma}{\sqrt{n}}, \mu + 3\frac{\sigma}{\sqrt{n}}\right]$$

with probability 0.9973.

Example1 Suppose that a beer producer wishes to establish a control chart for the volume of beer in a bottle. It is desired to have 341 ml in a bottle and $\sigma = 5$ ml. A control chart in that case for samples of size $n = 5$ is $\left[341 - 3\frac{5}{\sqrt{5}}, 341 + 3\frac{5}{\sqrt{5}}\right] = [341 - 6.7, 341 + 6.7] = [334.3, 347.7]$

What if μ, σ are unknown?

In that case, we must take the time to establish the control chart. This is done by taking k samples of n observations each. The samples will yield k averages

$$\bar{x}_1, \bar{x}_2, ..., \bar{x}_k$$

Usually, $k = 25$ or more. Then we calculate

$$\begin{aligned} \bar{\bar{x}} &= \frac{\bar{x}_1 + \bar{x}_2 + ... + \bar{x}_k}{k} \\ \bar{\bar{\sigma}} &= \frac{\bar{\sigma}_1 + ... + \bar{\sigma}_k}{k} \end{aligned}$$

where each $\bar{\sigma}_i$ is the standard deviation estimate of the ith sample, $i = 1, ..., k$. The new control chart is given by

$$\left[\bar{\bar{x}} - 3\frac{\bar{\bar{\sigma}}}{c\sqrt{n}}, \bar{\bar{x}} + 3\frac{\bar{\bar{\sigma}}}{c\sqrt{n}}\right]$$

where the constant c is chosen to provide an unbiased estimate of σ. Values of c for different sample sizes are given below

n	2	3	4	5	6	7	8	9	10
c	0.5642	0.7236	0.7979	0.8407	0.8686	0.8888	0.9027	0.9139	0.9227

3 Control Chart for the standard deviation

It is also possible to construct a control chart for the standard deviation σ. It can be theoretically shown that the expected value of the sample standard deviation for a sample of size n is $c\sigma$ while the variance is $\left[2(n-1)-2nc^2\right]\sigma/\sqrt{2n}$. Consequently the upper and lower control limits become

$$UCL = B_2\sigma, LCL = B_1\sigma$$

where

$$\begin{aligned} B_2 &= c + \frac{3}{\sqrt{2n}}\left[2(n-1)-2nc^2\right] \\ B_1 &= c - \frac{3}{\sqrt{2n}}\left[2(n-1)-2nc^2\right] \end{aligned}$$

Values of B_1, B_2 for different values of n are given below

n	2	3	4	5	6	7	8	9	10
B_1	0	0	0	0	.026	.105	.167	.219	.262
B_2	1.843	1.858	1.808	1.756	1.711	1.672	1.638	1.609	1.584

As before, if σ is unknown, we must take k samples of n observations each. Then we calculate

$$\bar{\bar{\sigma}} = \frac{\bar{\sigma}_1 + ... + \bar{\sigma}_k}{k}$$

where each $\bar{\sigma}_i$ is the standard deviation estimate of the ith sample, $i = 1, ..., k$ and estimate σ by $\frac{\bar{\bar{\sigma}}}{c}$ which becomes the new center line. The new control chart has upper and lower control limits

$$UCL = B_4\bar{\bar{\sigma}}, LCL = B_3\bar{\bar{\sigma}}$$

where

$$\begin{aligned} B_4 &= 1 + \frac{3}{\sqrt{2nc}}\left[2(n-1) - 2nc^2\right] \\ B_3 &= 1 - \frac{3}{\sqrt{2nc}}\left[2(n-1) - 2nc^2\right] \end{aligned}$$

Example To establish a control chart for the mean volume of beer in standard bottles, we collect $k = 25$ samples of 6 bottles each. We observe $\bar{\bar{x}} = 340.96$ and $\bar{\bar{\sigma}} = 4.764$. The control chart is then

$$340.96 \pm \frac{3(4.764)}{0.8686\sqrt{6}} = 340.96 \pm 6.717$$

A control chart for the standard deviation is $(B_3\bar{\bar{\sigma}}, B_4\bar{\bar{\sigma}}) = (0.03(4.764), 1.97(4.764)) = (0.143, 9.385)$

Clearly the values observed are between the control lines.

Suppose at a later time we have a sample of size 6 beer bottles and we observe $\bar{x} = 340.08$,$\bar{\sigma} = 4.764$.

4 Control Chart for percentage defective

Suppose that we would like to establish a control chart for the fraction of defective machine bolts. We take 25 subgroups of 50 bolts each. In each subgroup we count the number of defectives, denoted D, the proportion defectives p for each subgroup, and we observe the following data.

Subgroup	D	p	Subgroup	D	p	Subgroup	D	p
1	1	0.02	9	1	0.02	17	1	0.02
2	2	0.04	10	0	0.00	18	0	0.00
3	5	0.10	11	0	0.00	19	0	0.00
4	6	0.12	12	1	0.02	20	1	0.02
5	3	0.06	13	0	0.00	21	1	0.02
6	5	0.10	14	1	0.02	22	0	0.00
7	2	0.04	15	0	0.00	23	0	0.00
8	1	0.02	16	2	0.04	24	1	0.02
						25	0	0.00

We calculate in general that if n is the size of the subgroup and k is the number of subgroups we have

$$\bar{p} = \frac{Total\#defectives}{kn}$$

$$UCL = \bar{p} + 3\frac{\sqrt{\bar{p}\left(1-\bar{p}\right)}}{\sqrt{n}}$$

$$LCL = \bar{p} - 3\frac{\sqrt{\bar{p}\,(1-\bar{p})}}{\sqrt{n}}$$

If the $LCL < 0$, set $LCL = 0$.

In the specific example, we have

$$\bar{p} = \frac{34}{1250} = 0.0272$$

$$UCL = 0.0272 + 3\frac{\sqrt{0.0272\,(0.9728)}}{\sqrt{50}} = 0.0963$$

$$LCL = 0$$